

Zeta Functions for Elliptic Curves

I. Counting Bundles

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Abstract: To count bundles on curves, we study zetas of elliptic curves and their zeros. There are two types, i.e., the pure non-abelian zetas defined using moduli spaces of semi-stable bundles, and the group zetas defined for special linear groups. In lower ranks, we show that these two types of zetas coincide and satisfy the Riemann Hypothesis. For general cases, exposed is an intrinsic relation on automorphism groups of semi-stable bundles over elliptic curves, the so-called counting miracle. All this, together with Harder-Narasimhan, Desale-Ramanan and Zagier's result, gives an effective way to count semi-stable bundles on elliptic curves not only in terms of automorphism groups but more essentially in terms of their h^0 's. Distributions of zeros of high rank zetas are also discussed.

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1 High Rank Zetas for Elliptic Curves

1.1 Fat Moduli Spaces

Let X be an irreducible, reduced, regular projective curve of genus g defined over \mathbb{F}_q . Denote by $\mathcal{M}_{X,r}(d)$ the moduli space of rank r semi-stable bundles of degree d consisting of the Seshadri Jordan-Hölder equivalences of \mathbb{F}_q -rational semi-stable bundles. For our purpose, we consider $\mathcal{M}_{X,r}(d)$ in the sense of the *fat* moduli, meaning that ordinary moduli spaces equipped with the additional structure at Seshadri class $[\mathcal{E}]$ defined by the collection of semi-stable bundles in $[\mathcal{E}]$, namely, the set $\{\mathcal{E} : \mathcal{E} \in [\mathcal{E}]\}$ is adding at the point $[\mathcal{E}]$. $\mathcal{M}_{X,r}(d)$ equipped with such a structure is called a fat moduli space and denoted as $\mathbf{M}_{X,r}(d)$.

A natural question is to count these \mathbb{F}_q -rational semi-stable bundles \mathcal{E} . For this purpose, two invariants, namely, the automorphism group $\text{Aut}(X, \mathcal{E})$ and its global sections $h^0(X, \mathcal{E})$ can be naturally used. This then leads to the refined Brill-Noether loci

$$W_{X,r}^i(d) := \left\{ [\mathcal{E}] \in \mathbf{M}_{X,r}(d) : \min_{\mathcal{E} \in [\mathcal{E}]} h^0(X, \mathcal{E}) \geq i \right\}$$

and

$$[\mathcal{E}]^j := \{\mathcal{E} \in [\mathcal{E}] : \dim_{\mathbb{F}_q} \text{Aut} \mathcal{E} \geq j\}.$$

Recall that there exist natural isomorphisms

$$\mathbf{M}_{X,r}(d) \rightarrow \mathbf{M}_{X,r}(d + rm), \quad \mathcal{E} \mapsto A^m \otimes \mathcal{E}$$

and

$$\mathbf{M}_{X,r}(d) \rightarrow \mathbf{M}_{X,r}(-d + r(2g - 2)), \quad \mathcal{E} \mapsto K_X \otimes \mathcal{E}^\vee,$$

where A is an Artin line bundle of degree one and K_X denotes the dualizing bundle of X/\mathbb{F}_q . So we only need to count $\mathbf{M}_{X,r}(d_0)$ for $d_0 = 0, 1, \dots, r(g - 1)$. Accordingly, we set

$$\alpha_{X,r}(d) := \sum_{\mathcal{E} \in \mathbf{M}_{X,r}(d)} \frac{q^{h^0(X, \mathcal{E})} - 1}{\#\text{Aut}(\mathcal{E})}, \quad \beta_{X,r}(d) := \sum_{\mathcal{E} \in \mathbf{M}_{X,r}(d)} \frac{1}{\#\text{Aut}(\mathcal{E})}$$

with β a classical invariant ([HN]).

So, to count bundles, the problem becomes how to control $\alpha_{X,r}(d_0)$'s with d_0 ranging as above, and $\beta_{X,r}(q)$ with $q = 0, 1, \dots, r - 1$. For α , two general principles can be used for counting semi-stable bundles, namely, the vanishing theorem claiming that $h^1(X, \mathcal{E}) = 0$ if $d(\mathcal{E}) \geq r(2g - 2) + 1$ and the Clifford lemma claiming that $h^0(X, \mathcal{E}) \leq r + \frac{d}{2}$ if $0 \leq \mu(\mathcal{E}) \leq 2g - 2$. But this is merely the starting point. By contrasting, the invariant β can be understood, thanks to the high profile works of Harder-Narasimhan ([HN]), Desale-Ramanan ([DR]) and Zagier ([Z]).

To state it, let

$$\zeta_X(s) := \frac{\prod_{i=1}^{2g} (1 - \omega_i q^{-s})}{(1 - q^{-s})(1 - qq^{-s})}$$

be the Artin zeta function of X/\mathbb{F}_q ,

$$v_r(\mathbb{F}_q) := v_r := \frac{\prod_{i=1}^{2g} (1 - \omega_i)}{q - 1} q^{(r^2-1)(g-1)} \zeta_X(2) \cdots \zeta_X(r)$$

and

$$c_{r,d}(q) := c_{r,d} := \prod_{i=1}^{s-1} \frac{q^{(n_i+n_{i+1})\{n_1+\cdots+n_i\}d/n}}{1 - q^{n_i+n_{i+1}}}.$$

With above, then the above works mentioned can be strengthen as follows:

Theorem 1. ([Z, Thm 2]) For any pair (r, d) , we have

$$\beta_{X,r}(d) = \sum_{n_1, \dots, n_s > 0, \sum n_i = r} q^{(g-1) \sum_{i < j} n_i n_j} c_{r,d}(q) \prod_{i=1}^s v_{n_i}(\mathbb{F}_q).$$

1.2 Definition

Let E be a regular, integral projective elliptic curve defined over \mathbb{F}_q . Define rank r pure zeta function for E/\mathbb{F}_q by

$$\widehat{\zeta}_{E,r}(s) := \zeta_{E,r}(s) := \sum_{m=0}^{\infty} \sum_{V \in \mathbf{M}_{E,r}(d), d=rm} \frac{q^{h^0(C,V)} - 1}{\#\text{Aut}(V)} \cdot (q^{-s})^{d(V)}.$$

Then, by the vanishing theorem for semi-stable bundles,

$$\begin{aligned} \zeta_{E,r}(s) &= \sum_{V \in \mathbf{M}_{E,r}(0)} \frac{q^{h^0(C,V)} - 1}{\#\text{Aut}(V)} + \sum_{m=1}^{\infty} \sum_{V \in \mathbf{M}_{E,r}(rm)} \frac{q^{h^0(C,V)} - 1}{\#\text{Aut}(V)} \cdot (q^{-s})^{d(V)} \\ &= \alpha_{E,r}(0) + \sum_{m=1}^{\infty} \sum_{V \in \mathbf{M}_{E,r}(rm)} \frac{q^{rm} - 1}{\#\text{Aut}(V)} \cdot (q^{-s})^{rm} \\ &= \alpha_{E,r}(0) + \beta_{E,r}(0) \sum_{m=1}^{\infty} (q^{rm} - 1) \cdot (q^{-s})^{rm} \\ &= \alpha_{E,r}(0) + \beta_{X,r}(0) \cdot \left(\frac{(qt)^r}{1 - (qt)^r} - \frac{t^r}{1 - t^r} \right) \end{aligned}$$

Consequently,

$$\widehat{Z}_{E,r}\left(\frac{1}{qt}\right) = \widehat{Z}_{E,r}(t) = \alpha_{X,r}(0) + \beta_{E,r}(0) \cdot \frac{(Q-1)T}{(1-T)(1-QT)}.$$

Here, $T := t^r$, $Q = q^r$, $Z_{E,r}(t) = \zeta_{E,r}(s)$, and $\widehat{Z}_{E,r}(t) = \widehat{\zeta}_{E,r}(s)$. This then completes the proof of the following

Theorem 2. (i) $\zeta_{E,1}(s) = \zeta_E(s)$, the Artin zeta function for E/\mathbb{F}_q ;
(ii) (**Rationality**) There exists a degree 2 polynomial $P_{E,r}(T) \in \mathbb{Q}[T]$ of T such that

$$Z_{E,r}(t) = \frac{P_{E,r}(T)}{(1-T)(1-QT)} \quad \text{with} \quad T = t^r, \quad Q = q^r;$$

(iii) (**Functional equation**)

$$\widehat{Z}_{E,r}\left(\frac{1}{qt}\right) = \widehat{Z}_{E,r}(t),$$

Remark. The pure zeta here, a new genuine one, is quite different from the zeta introduced in [W1]. The reason for the purity is that the zeta in [W1] does not satisfy the Riemann Hypothesis.

1.3 Rank Two

For rank two pure zeta, it suffices to calculate $\alpha_{E,2}(0)$ and $\beta_{E,2}(0)$. For β , by Harder-Narasimhan, Desale-Ramanan and Zagier's formula, i.e., Thm 1,

$$\beta_{E,2}(0) = \frac{N}{q-1} \left(1 + \frac{N}{q^2-1}\right).$$

Here, as usual, N denotes the number of \mathbb{F}_q -rational points of E . On the other hand, by the classification of Atiyah ([A]), over $\overline{\mathbb{F}_q}$, the graded bundle associated to a Jordan-Hölder filtration of a semi-stable bundle $V \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ is of the form $\text{Gr}(V \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}) = L_1 \oplus L_2$ with L_i degree zero line bundles, which may not really be defined over \mathbb{F}_q . Consequently, for \mathbb{F}_q -rational semi-stable bundles V of rank two, $h^0(E, V) \neq 0$ if and only if $V = \mathcal{O}_E \oplus L$ or $V = I_2$ with L a \mathbb{F}_q -rational line bundle of degree 0 and I_2 the only non-trivial extension of \mathcal{O}_E by itself. Thus $\alpha_{E,2}(0)$ is given by

$$\begin{aligned} & \left(\frac{q^{h^0(E, \mathcal{O}_E \oplus \mathcal{O}_E)} - 1}{\#\text{Aut}(\mathcal{O}_E \oplus \mathcal{O}_E)} + \frac{q^{h^0(E, I_2)} - 1}{\#\text{Aut}(I_2)} \right) + \sum_{L \in \text{Pic}^0(E), L \neq \mathcal{O}_E} \frac{q^{h^0(E, \mathcal{O}_E \oplus L)} - 1}{\#\text{Aut}(\mathcal{O}_E \oplus L)} \\ &= \left(\frac{q^2 - 1}{(q^2 - 1)(q^2 - q)} + \frac{q - 1}{(q - 1)q} \right) + (N - 1) \frac{q - 1}{(q - 1)^2} = \frac{N}{q - 1}. \end{aligned}$$

Thus, we have the following

Proposition 3. (i) $\alpha_{E,2}(0) = \beta_{E,1}(0) = \frac{N}{q-1}$;

(ii) $Z_{E,2}(t) = \alpha_{E,2}(0) \cdot \frac{1 + (N-2)T + QT^2}{(1-T)(1-QT)}$.

The Riemann Hypothesis holds since

$$\Delta_2 = (N - 2)^2 - 4Q = (N - 2 - 2q)(N - 2 + 2q) < 0$$

using Hasse's theorem for the Riemann Hypothesis of elliptic curves, namely $N \leq 2\sqrt{q}$. That is to say, we have proved the following

Theorem 4. (Riemann Hypothesis₂)

$$\widehat{\zeta}_{E,2}(s) = 0 \quad \Rightarrow \quad \operatorname{Re}(s) = \frac{1}{2}.$$

1.4 Rank Three

To understand rank 3 zeta for elliptic curves, we need to calculate $\alpha_{E,3}(0)$. Recall that for semi-stable vector bundles V over elliptic curves, by Atiyah ([A]), the Jordan-Hölder graded bundles $G(V)$ are of the form $L_1 \oplus L_2 \oplus L_3$ with $L_i \in \operatorname{Pic}^0(E \otimes \overline{\mathbb{F}}_q)$. In particular, for semi-stable bundles with non-trivial contribution to $\alpha_{E,3}(0)$, at least one of the L_i 's should be \mathcal{O}_E . So three types:

- (1) $L_1 = L_2 = L_3 = \mathcal{O}_E$;
- (2) $L_1 = L_2 = \mathcal{O}_E \neq L_3$;
- (3) $L_1 = \mathcal{O}_E \neq L_i, i = 2, 3$.

Accordingly, write

$$\alpha_{E,3}(0) = \alpha_{E,3}^1(0) + \alpha_{E,3}^2(0) + \alpha_{E,3}^3(0),$$

with

$$\alpha_{E,3}^i(0) = \sum_{\substack{V \in \mathbf{M}_{E,3}(0), \\ V: \text{type } i}} \frac{q^{h^0(X,V)} - 1}{\#\operatorname{Aut}(V)}, \quad i = 1, 2, 3.$$

Clearly, for types (1) and (2), all L_i 's are \mathbb{F}_q -rational. However, for type (3), it may well be possible that L_2 and L_3 are not, even V itself is \mathbb{F}_q -rational. For general curves, this proves to be an essential difficulty in understanding rank 3 zetas. However, for elliptic curves, due to the degree constraint, we find a nice way to overcome it.

The idea is to use Thm 1. To be more precise, for type (3), we have

$$\begin{aligned} \alpha_{E,3}^3(0) &= \sum_{\substack{V \in \mathbf{M}_{E,3}(0), \\ G(V) = \mathcal{O}_E \oplus L_2 \oplus L_3 \\ L_2 \neq \mathcal{O}_E \neq L_3}} \frac{q^1 - 1}{\#\operatorname{Aut}(V)} \\ &= \sum_{\substack{W \in \mathbf{M}_{E,2}(0), \\ G(W) = L_2 \oplus L_3 \\ L_2 \neq \mathcal{O}_E \neq L_3}} \frac{q - 1}{\#\operatorname{Aut}(W) \cdot (q - 1)} = \sum_{\substack{W \in \mathbf{M}_{E,2}(0), \\ G(W) = L_2 \oplus L_3 \\ L_2 \neq \mathcal{O}_E \neq L_3}} \frac{1}{\#\operatorname{Aut}(W)}, \end{aligned}$$

since there is no non-zero morphisms between \mathcal{O}_E and L_i 's. But

$$\begin{aligned} & \left\{ W \in \mathbf{M}_{E,2}(0) : G(W) = L_2 \oplus L_3, L_2 \neq \mathcal{O}_E \neq L_3 \right\} \\ &= \mathbf{M}_{E,2}(0) \setminus \left\{ W \in \mathbf{M}_{E,2}(0) : G(W) = \mathcal{O}_E \oplus L \right\}. \end{aligned}$$

Consequently,

$$\begin{aligned} \alpha_{E,3}^3(0) &= \beta_{E,2}(0) - \sum_{\substack{W \in \mathbf{M}_{E,2}(0), \\ G(W) = \mathcal{O}_E \oplus L}} \frac{1}{\#\text{Aut}(W)} \\ &= \beta_{E,2}(0) - \sum_{\substack{W \in \mathbf{M}_{E,2}(0), \\ G(W) = \mathcal{O}_E \oplus L, L \neq \mathcal{O}_E}} \frac{1}{\#\text{Aut}(W)} - \left(\frac{1}{\#\text{Aut}(\mathcal{O}_E \oplus \mathcal{O}_E)} + \frac{1}{\#\text{Aut}(I_2)} \right) \\ &= \beta_{E,2}(0) - \sum_{W = \mathcal{O}_E \oplus L, L/\mathbb{F}_q \neq \mathcal{O}_E} \frac{1}{\#\text{Aut}(W)} - \left(\frac{1}{(q^2-1)(q^2-q)} + \frac{1}{(q-1)q} \right) \\ &= \beta_{E,2}(0) - \frac{N-1}{(q-1)^2} - \frac{q}{(q^2-1)(q-1)} \\ &= \frac{N^2 + (q^2 - q - 2)N + 1}{(q^2-1)(q-1)}. \end{aligned}$$

Now by Thm 1, we have

$$\begin{aligned} \beta_{E,2}(0) &= \frac{N}{q-1} \zeta_E(2) + \frac{1}{1-q^2} \left(\frac{N}{q-1} \right)^2 \\ &= \frac{N}{q-1} \left(1 + \frac{N}{q^2-1} \right). \end{aligned}$$

For type (2), we have

$$\begin{aligned} \alpha_{E,3}^2(0) &= \sum_{L/\mathbb{F}_q \neq \mathcal{O}_E} \left(\frac{q^2-1}{\#\text{Aut}(\mathcal{O}_E \oplus \mathcal{O}_E \oplus L)} + \frac{q-1}{\#\text{Aut}(I_2 \oplus L)} \right) \\ &= (N-1) \left(\frac{q^2-1}{(q^2-1)(q^2-q)(q-1)} + \frac{q-1}{(q^2-q)(q-1)} \right) \\ &= \frac{q}{(q^2-1)(q-1)}. \end{aligned}$$

Finally for type (1), we have

$$\begin{aligned} \alpha_{E,3}^1(0) &= \frac{q^3-1}{\#\text{Aut}(\mathcal{O}_E \oplus \mathcal{O}_E \oplus \mathcal{O}_E)} + \frac{q^2-1}{\#\text{Aut}(I_2 \oplus \mathcal{O}_E)} + \frac{q-1}{\#\text{Aut}(I_3)} \\ &= \frac{q^3-1}{(q^3-1)(q^3-q)(q^3-q^2)} + \frac{q^2-1}{(q-1)^2 q^3} + \frac{q-1}{(q-1)q^2} \\ &= \frac{N-1}{(q-1)^2}. \end{aligned}$$

Put all this together, we have

Proposition 5. (i) $\alpha_{E,3}(0) = \beta_{E,2}(0) = \frac{N}{q-1} \cdot \left(1 + \frac{N}{q^2-1}\right)$;
(ii) $\beta_{E,3}(0) = \frac{N}{q-1} \cdot \left[1 + \frac{q+2}{q^3-1}N + \frac{N^2}{(q^3-1)(q^2-1)}\right]$;
(iii) $\widehat{Z}_{E,3}(t) = \frac{N}{q-1} \cdot \frac{1}{(1-T)(1-q^3T)}$
 $\times \left[\left(1 + \frac{N}{q^2-1}\right)(1 + q^3T^2) + \left(-2 + \frac{(2q-3)N}{q-1} + \frac{N^2}{q^2-1}\right) \cdot T\right]$.

Proof. For our convenience, set $\widehat{\zeta}_E(1) = \frac{N}{q-1}$. Then for (ii), by Thm 1 and $\widehat{\zeta}_E(3) = \widehat{\zeta}_E(1) \cdot \left(1 + \frac{q^2N}{(q^3-1)(q^2-1)}\right)$, we have

$$\begin{aligned} \beta_{E,3}(0) &= \widehat{\zeta}_E(1)\widehat{\zeta}_E(2)\widehat{\zeta}_E(3) - \frac{2}{1-q^3}\widehat{\zeta}_E(1)\widehat{\zeta}_E(1)\widehat{\zeta}_E(2) \\ &\quad + \frac{1}{(1-q^2)(1-q^2)}\widehat{\zeta}_E(1)\widehat{\zeta}_E(1)\widehat{\zeta}_E(1) \\ &= \widehat{\zeta}_E(1) \left[1 + \frac{q+2}{q^3-1}N + \frac{N^2}{(q^3-1)(q^2-1)}\right]. \end{aligned}$$

For (iii), from the expression of $\zeta_{E,r}(s)$ obtained in Thm. 2,

$$\begin{aligned} \widehat{Z}_{E,3}(t) &= \frac{N}{q-1} \\ &\quad \times \left[\left(1 + \frac{N}{q^2-1}\right) + \left(1 + \frac{q+2}{q^3-1}N + \frac{N^2}{(q^3-1)(q^2-1)}\right) \frac{(q^3-1)T}{(1-T)(1-q^3T)}\right] \\ &= \frac{\widehat{\zeta}_E(1)}{(1-T)(1-q^3T)} \\ &\quad \times \left[\left(1 + \frac{N}{q^2-1}\right)(1 + q^3T^2) + \left(-2 + \frac{(2q-3)N}{q-1} + \frac{N^2}{q^2-1}\right)T\right] \end{aligned}$$

2 Zetas of Elliptic Curves associated to SL_n

2.1 Definition

For $G = SL_n$ with B the standard Borel subgroup consisting of upper triangular matrices, let T be the associated torus consisting of diagonal matrices. Then the root system Φ associated to T can be realized as

$$\Phi^+ = \{e_i - e_j : 1 \leq i < j \leq n\}$$

with $\{e_i\}_{i=1}^n$ the standard orthogonal basis of the Euclidean space $V = \mathbb{R}^n$. Being type A_{n-1} , its simple roots are given by

$$\Delta := \{\alpha_i := e_i - e_{i+1} : i = 1, 2, \dots, n-1\},$$

the so-called Weyl vector is simply

$$\rho := \frac{1}{2} \left((n-1)e_1 + (n-3)e_2 + \dots - (n-3)e_{n-1} - (n-1)e_n \right),$$

and the Weyl group W may be identified with the permutation group S_n via the action on the subindex of e_i 's. Introduce the corresponding fundamental weights λ_j 's via

$$\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}, \quad \forall \alpha_j \in \Delta.$$

For each $w \in W$, set $\Phi_w := \Phi^+ \cap w^{-1}\Phi^-$. For $\lambda \in V_{\mathbb{C}}$, introduce then the period of SL_n by

$$\omega_E^{SL_n}(\lambda) := \sum_{w \in W} \frac{1}{\prod_{\alpha \in \Delta} (1 - q^{-\langle w\lambda - \rho, \alpha^\vee \rangle})} \prod_{\alpha \in \Phi_w} \frac{\widehat{\zeta}_E(\langle \lambda, \alpha^\vee \rangle)}{\widehat{\zeta}_E(\langle \lambda, \alpha^\vee \rangle + 1)}.$$

Corresponding to $\alpha_P = \alpha_{n-1}$, let

$$P = P_{n-1,1} = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : A \in GL_{n-1}, D \in GL_1 \right\}$$

be the standard parabolic subgroup of SL_n attached to the partition $n = (n-1) + 1$. Write

$$\lambda := \rho + \sum_{j=1}^{n-1} s_j \lambda_j$$

and set $s := s_{n-1}$. Then we introduce the *period for* (SL_n, P) as a one variable function defined by

$$\omega_E^{SL_n/P}(s) := \text{Res}_{s_1=0} \text{Res}_{s_2=0} \cdots \text{Res}_{s_{n-2}=0} \omega_E^{SL_n}(\lambda).$$

This period consists many terms, each of which is a product of certain rational factors of q^{-s} and Atrin zetas. Clear up all zeta factors in the denominators of all terms! The resulting function is then defined to be the *zeta function* $\widehat{\zeta}_E^{SL_n}(s)$ of E associated to SL_n .

Theorem 6. ($[W_4]$) (i) $\widehat{\zeta}_E^{SL_n}(s)$ is a well-defined meromorphic function on the whole s -plane;
(ii) (**Functional Equation**)

$$\widehat{\zeta}_E^{SL_n}(-n-s) = \widehat{\zeta}_E^{SL_n}(s).$$

This group theoretic zeta function is expected to play a central role in counting bundles. In fact, we have the following

Conjecture 7. (i) (**The Riemann Hypothesis**)

$$\widehat{\zeta}_E^{SL_n}(s) = 0 \quad \Rightarrow \quad \text{Re}(s) = -\frac{n}{2}.$$

(ii) (**Uniformity**) Up to a rational function factor of q ,

$$\widehat{\zeta}_E^{SL_n}(-ns) = \widehat{\zeta}_{E,r}(s).$$

In the later discussion, for our own convenience, we will freely make linear changes of the variables for $\widehat{\zeta}_E^{SL_n}(s)$ and denote the resulting functions by $\widehat{\zeta}_E^{SL_n}(s)$ as well.

2.2 SL_2

By definition, a direct calculation (with a linear change of variable) leads to,

$$\widehat{\zeta}_E^{SL_2}(s) := \frac{\widehat{\zeta}_E(2s)}{1 - q^{-2s+2}} + \frac{\widehat{\zeta}_E(2s-1)}{1 - q^{2s}}.$$

Set $t = q^{-s}$, $T = t^2$ and $a_1 = q + 1 - N$. Then

$$\begin{aligned} \widehat{\zeta}_E^{SL_2}(t) &= \frac{1 - a_1T + qT^2}{(1-T)(1-qT)(1-q^2T)} + \frac{1 - a_1qT + q^3T^2}{(1-qT)(1-q^2T^2)(1-\frac{1}{T})} \\ &= \frac{1 - a_1T + qT^2 - T(1 - a_1qT + q^3T^2)}{(1-T)(1-qT)(1-q^2T)}. \end{aligned}$$

Consequently,

$$\widehat{\zeta}_E^{SL_2}(s) = \frac{1 + (N-2)T + q^2T^2}{(1-T)(1-q^2T)}.$$

Theorem 8. *Conjecture 7 holds for SL_2 . That is to say, the Uniformity and the Riemann Hypothesis hold for $\widehat{\zeta}_E^{SL_2}(s)$ and $\widehat{\zeta}_{E,2}(s)$.*

Remark. Yoshida ([Y]) shows that, more generally, $\widehat{\zeta}_X^{SL_2}(s)$ satisfies the RH for all regular, integral, projective curve X defined over \mathbb{F}_q .

2.3 SL_3

By definition, a direct calculation (with a linear change of variable) shows that the zeta of E associated to SL_3 is given by

$$\begin{aligned} \widehat{\zeta}_E^{SL_3}(s) &= \widehat{\zeta}_E(2) \cdot \left(\frac{\widehat{\zeta}_E(3s)}{1 - q^{-3s+3}} + \frac{\widehat{\zeta}_E(3s-2)}{1 - q^{3s}} \right) \\ &\quad + \frac{\widehat{\zeta}_E(1)}{1 - q^2} \cdot \left(\frac{\widehat{\zeta}_E(3s)}{1 - q^{-3s+2}} + \frac{\widehat{\zeta}_E(3s-2)}{1 - q^{3s-1}} \right) \\ &\quad + \widehat{\zeta}_E(1) \cdot \frac{\widehat{\zeta}_E(3s-1)}{(1 - q^{3s})(1 - q^{-3s+3})}. \end{aligned}$$

With $t = q^{-s}$, $T = t^3$, we have

$$\widehat{\zeta}_E^{SL_3}(s) = \frac{P_E^{SL_3}(T)}{(1-T)(1-qT)(1-q^2T)(1-q^3T)}$$

with

$$\begin{aligned}
P_E^{SL_3}(T) &:= \left(1 + \frac{qN}{(q-1)(q^2-1)}\right) \\
&\times \left[(1 + q^6T^4 - (q^2 + q + 2 - N)(T + q^3T^3) + 2qT^2(1 + (q+1-N)q)) \right. \\
&\quad \left. - \frac{N}{(q-1)(q^2-1)} \right] \\
&\times \left[1 + q^6T^4 - (q^3 + 2q + 1 - N)(T + q^3T^3) + 2qT^2(1 + (q^2(q+1-N))) \right] \\
&\quad - \frac{N}{q-1} \left[(T + q^3T^3) - (q+1-N)qT^2 \right].
\end{aligned}$$

Lemma 9. (i) *There exists a degree 2 polynomial $P_{E,o}^{SL_3}(T)$ of T such that*

$$P_E^{SL_3}(T) = (1 - qT)(1 - q^2T)P_{E,o}^{SL_3}(T);$$

(ii) $P_{E,o}^{SL_3}(T)$ is given by

$$P_{E,o}^{SL_3}(T) = \left(1 + \frac{N}{q^2-1}\right) + \left[-2 + \frac{2q-3}{q-1}N + \frac{N^2}{q^2-1}\right] \cdot T + \left(1 + \frac{N}{q^2-1}\right)q^3T^2.$$

Proof. (i) By functional equation, it suffices to show that

$$P_E^{SL_3}\left(\frac{1}{q}\right) = 0.$$

Then routine checking. In fact, one can first set $qT = 1$ in the above expression of $P_E^{SL_3}(T)$. Then using SIMPLIFY command of Mathematica, to verify that the resulting complicated combination in terms of q and N gives us 0 as wanted.

(ii) You can directly calculate it by hands. Instead, first we have

$$\begin{aligned}
P_E^{SL_3}(T) &:= \left[1 + \frac{N}{q^2-1}\right] \cdot (1 + q^6T^4) \\
&\times \left[-(q^2 + q + 2) + N\frac{q-3}{q-1} + \frac{N^2}{q^2-1} \right] \cdot (T + q^3T^3) \\
&\left[2(q^2 + q + 1) - N\frac{2q^3 - q^2 - 4q - 3}{q^2-1} - \frac{N^2}{q-1} \right] \cdot qT^2.
\end{aligned}$$

Then using the PolynomialQuotientRemainder command of Mathematica, to divide $P_E^{SL_3}(T)$ by $q^3T^2 - (q^2 + q)T + 1$. As a result, Mathematica would give us

(a) the quotient, a degree two polynomial of T with coefficients in terms of q and N . Using Simplify command of Mathematica to get the result in the lemma;

(b) the reminder, a linear polynomial in T with coefficients in terms of q and N . Using Simplify command of Mathematica to see that the coefficients of T and constant terms are all 0.

This then completes the proof of the lemma.

Theorem 10. (i) (**Uniformity**₃)

$$\widehat{\zeta}_{E,3}(s) = \widehat{\zeta}_E(1) \cdot \widehat{\zeta}_E^{SL_3}(s).$$

(ii) (**Riemann Hypothesis**₃)

$$\widehat{\zeta}_{E,3}(s) = 0 \quad \Rightarrow \quad \operatorname{Re}(s) = \frac{1}{2}.$$

Proof. (i) is a direct consequence of the closed formulas for $\widehat{\zeta}_{E,2}(s)$ and $\widehat{\zeta}_E^{SL_3}(s)$ in Prop 3 and Lem 9.

For (ii), it suffices to show that the discriminant of the degree two polynomial $P_{E,o}^{SL_3}(T)$ is strictly negative. Clearly,

$$\begin{aligned} \Delta_3 &= \left[-2 + \frac{2q-3}{q-1}N + \frac{N^2}{q^2-1} \right]^2 - 4 \left(1 + \frac{N}{q^2-1} \right)^2 q^3 \\ &= \left[\left(-2 + \frac{2q-3}{q-1}N + \frac{N^2}{q^2-1} \right) + 2 \left(1 + \frac{N}{q^2-1} \right) q\sqrt{q} \right] \\ &\quad \times \left[\left(-2 + \frac{2q-3}{q-1}N + \frac{N^2}{q^2-1} \right) - 2 \left(1 + \frac{N}{q^2-1} \right) q\sqrt{q} \right]. \end{aligned}$$

The first factor is strictly positive, while by Hasse's theorem for Artin zetas, the second factor is strictly negative. This then completes the proof.

2.4 Distribution of zeros

For Artin zetas of elliptic curves E/\mathbb{F}_q , set

$$\cos \theta_p = \frac{p+1 - N(E/\mathbb{F}_p)}{2\sqrt{p}}, \quad 0 < \theta_p < \pi.$$

Conjecture 11. (Sato-Tate Conjecture) *If $\{E/\mathbb{F}_p\}$ are not (resulting from a global one of CM type), then for $0 \leq \alpha < \beta \leq \pi$,*

$$\lim_{x \rightarrow \infty} \frac{\#\{p \text{ prime} : p \leq x, \alpha \leq \theta_p \leq \beta\}}{\#\{p \text{ prime} : p \leq x\}} = \frac{2}{\pi} \int_{\alpha}^{\beta} \sin^2 \theta d\theta.$$

There are some exciting developments in this direction due to Taylor, Clozel-Harris-Shepherd-Barron and Barnet-Lamb-Geraghty-Harris.

Motivated by this, by the RH for SL_2 and SL_3 , we set

$$\cos \theta_{2,p} = \frac{N(E/\mathbb{F}_p) - 2}{2p},$$

$$\cos \theta_{3,p} = \frac{-2 + \frac{2p-3}{p-1}N(E/\mathbb{F}_p) + \frac{N(E/\mathbb{F}_p)^2}{p^2-1}}{2p\sqrt{p} \cdot \left(1 + \frac{N(E/\mathbb{F}_p)}{p^2-1}\right)},$$

and hence to understand

$$\lim_{x \rightarrow \infty} \frac{\#\{p \text{ prime} : p \leq x, \alpha \leq \theta_{2,p} \leq \beta\}}{\#\{p \text{ prime} : p \leq x\}}$$

and

$$\lim_{x \rightarrow \infty} \frac{\#\{p \text{ prime} : p \leq x, \alpha \leq \theta_{3,p} \leq \beta\}}{\#\{p \text{ prime} : p \leq x\}}.$$

By Hasse's theorem, we have $|N(E/\mathbb{F}_p) - p - 1| \leq 2\sqrt{p}$. Thus,

$$\lim_{p \rightarrow \infty} \frac{N(E/\mathbb{F}_p) - 2}{2p} = \frac{1}{2}$$

and

$$\lim_{p \rightarrow \infty} \frac{-2 + \frac{2p-3}{p-1}N(E/\mathbb{F}_p) + \frac{N(E/\mathbb{F}_p)^2}{p^2-1}}{2p\sqrt{p} \cdot \left(1 + \frac{N(E/\mathbb{F}_p)}{p^2-1}\right)} = 0.$$

Consequently,

$$\lim_{p \rightarrow \infty} \theta_{2,p} = \frac{\pi}{3}, \quad \lim_{p \rightarrow \infty} \theta_{3,p} = \frac{\pi}{2}.$$

Therefore, we have the following

Proposition 12. *The distributions of zeros for rank 2 zeta, resp. rank 3 zetas of elliptic curves are of Dirac type. More precisely,*

$$\lim_{x \rightarrow \infty} \frac{\#\{p \text{ prime} : p \leq x, \alpha \leq \theta_{2,p} \leq \beta\}}{\#\{p \text{ prime} : p \leq x\}} = \int_{\alpha}^{\beta} \delta_{\frac{\pi}{3}} dt$$

and

$$\lim_{x \rightarrow \infty} \frac{\#\{p \text{ prime} : p \leq x, \alpha \leq \theta_{3,p} \leq \beta\}}{\#\{p \text{ prime} : p \leq x\}} = \int_{\alpha}^{\beta} \delta_{\frac{\pi}{2}} dt,$$

where δ_a denotes the Dirac distribution at a .

3 Counting Bundles

Recall that the rank r pure non-abelian zeta function of an elliptic curve E/\mathbb{F}_q is given by

$$\widehat{Z}_{E,r}(t) = \alpha_{E,r}(0) + \beta_{E,r}(0) \cdot \frac{(Q-1)T}{(1-T)(1-QT)} = \frac{P_{E,r}(T)}{(1-T)(1-QT)}$$

with

$$P_{E,r}(T) = \alpha_{E,r}(0) - \left[(Q+1)\alpha_{E,r}(0) - (Q-1)\beta_{E,r}(0) \right] T + \alpha_{E,r}(0) QT^2.$$

Here $t = q^{-s}$, $Q = q^r$ and $T = t^r$. Thus to determine it, we need to know the invariants $\alpha_{E,r}(0)$ and $\beta_{E,r}(0)$.

As said, the β -invariant has been studied by many authors. In fact Harder-Narasimhan, Desale-Ramanan, and Zagier's formula can be arranged as follows:

Theorem 13.

$$\beta_{E,n}(0) = \sum_{n_1+\dots+n_k=n} \prod_{j=1}^{k-1} \frac{1}{q^{n_j+n_{j+1}} - 1} v_{n_1,\dots,n_k}$$

where

$$v_{n_1,\dots,n_k} := \prod_{j=1}^k v_{n_j} \quad \text{with} \quad v_n := \widehat{\zeta}_E(1)\widehat{\zeta}_E(2)\cdots\widehat{\zeta}_E(n).$$

This is significantly clearer than the original formula stated in Theorem 1, since the parabolic reduction structure appears and the parabolic coefficients

$$e_{n_1,\dots,n_k} = \prod_{j=1}^{k-1} \frac{1}{q^{n_j+n_{j+1}} - 1}$$

is *environmentally free*, i.e., only determined by the group structure but independent of curves. Indeed, for general genus curve X the same formula holds if we rewrite the left hand as

$$u_n := \frac{\beta_{X,n}(0)}{q^{\frac{n(n+1)}{2}(g-1)}}.$$

Theorem 14. (See e.g., [W5])

$$u_n = \sum_{n_1+\dots+n_k=n} (-1)^k e_{n_1,\dots,n_k} \cdot v_{n_1,\dots,n_k}.$$

As for the α -invariant, by our previous calculation in lower ranks, we introduce the following

Conjecture 15. (Counting Miracle) For elliptic curves E/\mathbb{F}_q ,

$$\alpha_{E,n+1}(0) = \beta_{E,n}(0)$$

We have checked it for $n = 1, 2, 3, 4, 5$.

To understand this, let us introduce the so-called Atiyah bundles I_r inductively. The starting point is $I_1 = \mathcal{O}_E$. Then we consider the extension \mathcal{O}_E by \mathcal{O}_E . Since $\text{Ext}_E^1(\mathcal{O}_E, \mathcal{O}_E)$ is 1 dimensional, there is, up to isomorphism, only one non-trivial extension. This is I_2 , namely, we have the non-trivial extension

$$0 \rightarrow \mathcal{O}_E \rightarrow I_2 \rightarrow \mathcal{O}_E \rightarrow 0.$$

Inductively, we know that $\text{Ext}_E^1(I_{r-1}, \mathcal{O}_E)$ is 1 dimensional, so there is, up to isomorphism, only one non-trivial extension of I_{r-1} by \mathcal{O}_E . This is I_r , namely, we have the non-trivial extension

$$0 \rightarrow \mathcal{O}_E \rightarrow I_r \rightarrow I_{r-1} \rightarrow 0.$$

Lemma 16. (i) ([A, Thm 8]) $h^0(E, I_r) = 1$ and

$$I_r \otimes I_s = I_{r-s+1} \oplus I_{r-s+3} \oplus \cdots \oplus I_{r+s}, \quad r \geq s;$$

(ii) For a partition $n = m_1 \cdot r_1 + m_2 \cdot r_2 + \cdots + m_s \cdot r_s$

$$= (r_1 + \cdots + r_1) + (r_2 + \cdots + r_2) + \cdots + (r_s + \cdots + r_s)$$

arranging in the order $r_1 < r_2 < \cdots < r_s$, we have

$$\begin{aligned} \#\text{Aut}\left(\bigoplus_{j=1}^s I_{r_j}^{\oplus m_j}\right) &= q^{2\sum_{1 \leq i < j \leq s} r_i m_i m_j} \\ &\times \prod_{j=1}^s (q^{m_j} - 1)(q^{m_j} - q) \cdots (q^{m_j} - q^{m_j-1}) q^{m_j^2(r_j-1)}. \end{aligned}$$

Proof. In (i), by definition, the first on h^0 is obvious, and the multiplicative relation is given in Thm 8 of [A]. As for (ii), we use the natural surjective morphism

$$\text{Aut}\left(\bigoplus_{j=1}^s I_{r_j}^{\oplus m_j}\right) \xrightarrow{\Pi} \prod_{j=1}^s \text{Aut}\left(I_{r_j}^{\oplus m_j}\right)$$

with kernel

$$\left[\text{Id} + \bigoplus_{i < j} \text{Hom}\left(I_{r_i}^{\oplus m_i}, I_{r_j}^{\oplus m_j}\right)\right] \times \left[\text{Id} + \bigoplus_{i < j} \text{Hom}\left(I_{r_j}^{\oplus m_j}, I_{r_i}^{\oplus m_i}\right)\right]$$

Note that

$$\text{Aut}(I_r) \simeq \left\{ \begin{pmatrix} a & b_1 & b_2 & \cdots & b_{r-2} & b_{r-1} \\ 0 & a & b_1 & \cdots & b_{r-3} & b_{r-2} \\ 0 & 0 & a & \cdots & b_{r-4} & b_{r-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a & b_1 \\ 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix} \in GL_r(\mathbb{F}_q) \right\}.$$

Hence

$$\#\text{Aut}(I_r) = (q-1)q^{r-1}.$$

More generally, for $\text{Aut}(I_r^{\oplus m_j})$, let us decompose its elements into $m_j \times m_j$ blocks of size $r_j \times r_j$. Then corresponding to the simple factor $q-1$ for I_r , now we have the factor $(q^{m_j}-1)(q^{m_j}-q^2)\cdots(q^{m_j}-q^{m_j-1})$, the number elements of $GL_{m_j}(\mathbb{F}_q)$, and the off diagonal parts of I_r now give us the total number $(q^{r_j-1})^{m_j^2}$ since there are $m_j \times m_j$ -blocks. Consequently, we have

$$\#\text{Aut}(I_r^{\oplus m_j}) = \left[(q^{m_j}-1)(q^{m_j}-q)\cdots(q^{m_j}-q^{m_j-1}) \right] \cdot q^{m_j^2(r_j-1)}.$$

Thus to complete the proof, we need to show that

$$\# \oplus_{i < j} \text{Hom}(I_{r_i}^{\oplus m_i}, I_{r_j}^{\oplus m_j}) = q^{\sum_{1 \leq i < j \leq s} r_i m_i m_j}.$$

This results from Atiyah's multiplicative structure on I_r . Indeed,

$$\begin{aligned} \text{Hom}(I_{r_i}^{\oplus m_i}, I_{r_j}^{\oplus m_j}) &\simeq H^0(E, I_{r_i}^\vee \otimes I_{r_j})^{\oplus m_i m_j} \\ &\simeq H^0(E, I_{r_i} \otimes I_{r_j})^{\oplus m_i m_j} \\ &\simeq H^0(E, I_{r_j-r_i+1} \oplus I_{r_j-r_i+3} \oplus \cdots \oplus I_{r_j+r_i-1})^{\oplus m_i m_j} \\ &= (\mathbb{F}_q)^{r_i m_i m_j}. \end{aligned}$$

This then completes the proof.

With this in mind, now we introduce the following

Conjecture 17. (Miracle of I_r)

$$\begin{aligned} \sum_{\sum_{i=1}^s r_i m_i = n+1} \frac{q^{h^0(E, \oplus_{j=1}^s I_{r_j}^{\oplus m_j})} - 1}{\#\text{Aut}(\oplus_{j=1}^s I_{r_j}^{\oplus m_j})} &= \sum_{\sum_{i=1}^s r_i m_i = n} \frac{1}{\#\text{Aut}(\oplus_{j=1}^s I_{r_j}^{\oplus m_j})} \\ &= \frac{q^{\frac{n(n+1)}{2}}}{(q^{n+1}-1)(q^n-1)(q^{n-1}-1)\cdots(q-1)}. \end{aligned}$$

Proposition 18. *Miracle of I_r implies the counting miracle.*

Proof. Let V be a semi-stable vector bundle of rank r over E/\mathbb{F}_q . Then the graded bundle associated to its Jordan-Hölder filtrations decomposes as

$$G(V) = L_1 \oplus L_2 \oplus \cdots \oplus L_r$$

with L_i 's line bundles of degree on $\overline{E} := E \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. As L_i need not be defined over \mathbb{F}_q , usually it is a bit complicated to classify V . As a matter of fact, this classification problem is related with arithmetic of elliptic curve, say, depending on the number of \mathbb{F}_q -rational r torsions of E . Instead of counting

them using a complete list, we first note that in order to have a non-trivial contribution to α , $h^0(V) \neq 0$. Guided by this, we regroup the summation as follows:

$$\alpha_{E,r}(0) = \sum_{i=1}^r \sum_{\substack{V, \\ G(V)=\mathcal{O}_E^{\oplus i} \oplus L_{i+1} \oplus \cdots \oplus L_r, \\ L_j \neq \mathcal{O}_E}} \frac{q^{h^0(E,V)} - 1}{\#\text{Aut} V}.$$

With $G(V) = \mathcal{O}_E^{\oplus i} \oplus L_{i+1} \oplus \cdots \oplus L_r$, $L_j \neq \mathcal{O}_E$, since there is no morphism between \mathcal{O}_E and L_j 's,

$$\text{Aut} V \simeq \text{Aut} U \times \text{Aut} W$$

with $G(U) = \mathcal{O}_E^{\oplus i}$ and $G(W) = L_{i+1} \oplus \cdots \oplus L_r$, $L_j \neq \mathcal{O}_E$. Consequently,

$$\begin{aligned} \alpha_{E,r}(0) &= \sum_{i=1}^r \sum_{\substack{V, V=U \oplus W, \\ G(U)=\mathcal{O}_E^{\oplus i}, \\ G(W)=L_{i+1} \oplus \cdots \oplus L_r, \\ L_j \neq \mathcal{O}_E}} \frac{q^{h^0(E,U)} - 1}{\#\text{Aut} U \cdot \#\text{Aut} W} \\ &= \sum_{i=1}^r \sum_{\substack{U, \\ G(U)=\mathcal{O}_E^{\oplus i}}} \frac{q^{h^0(E,U)} - 1}{\#\text{Aut} U} \sum_{\substack{W, \\ G(W)=L_{i+1} \oplus \cdots \oplus L_r, \\ L_j \neq \mathcal{O}_E}} \frac{1}{\#\text{Aut} W}. \end{aligned}$$

Now assume that we have the Miracle for I_r , then

$$\sum_{\substack{U, \\ G(U)=\mathcal{O}_E^{\oplus i}}} \frac{q^{h^0(E,U)} - 1}{\#\text{Aut} U} = \sum_{\substack{U, \\ G(U)=\mathcal{O}_E^{\oplus i-1}}} \frac{1}{\#\text{Aut} U}.$$

Therefore, by reversing the above discussion with $q^{h^0} - 1$ replaced by 1, we get

$$\begin{aligned} \alpha_{E,r}(0) &= \sum_{i=1}^r \sum_{\substack{U, \\ G(U)=\mathcal{O}_E^{\oplus i-1}}} \frac{1}{\#\text{Aut} U} \sum_{\substack{W, \\ G(W)=L_{i+1} \oplus \cdots \oplus L_r, \\ L_j \neq \mathcal{O}_E}} \frac{1}{\#\text{Aut} W} \\ &= \sum_{i=1}^r \sum_{\substack{V, V=U \oplus W, \\ G(U)=\mathcal{O}_E^{\oplus i-1}, \\ G(W)=L_{i+1} \oplus \cdots \oplus L_r, \\ L_j \neq \mathcal{O}_E}} \frac{1}{\#\text{Aut} U \cdot \#\text{Aut} W} \\ &= \sum_{i=1}^r \sum_{\substack{V, \\ G(V)=\mathcal{O}_E^{\oplus i-1} \oplus L_{i+1} \oplus \cdots \oplus L_r, \\ L_j \neq \mathcal{O}_E}} \frac{1}{\#\text{Aut} V} \\ &= \beta_{E,r-1}(0). \end{aligned}$$

4 Distribution of Zeros

4.1 The Riemann Hypothesis

For high rank pure zeta functions of elliptic curves, with our works on rank two and three zeta functions, we now introduce the following

Conjecture 19. (Riemann Hypothesis)

$$\widehat{\zeta}_{E,r}(s) = 0 \quad \Rightarrow \quad \operatorname{Re}(s) = \frac{1}{2}.$$

Since

$$\begin{aligned} \widehat{Z}_{E,r}(t) &= \alpha_{E,r}(0) + \beta_{E,r}(0) \cdot \frac{(Q-1)T}{(1-T)(1-QT)} \\ &= \frac{P_{E,r}(T)}{(1-T)(1-QT)} \end{aligned}$$

with

$$\begin{aligned} P_{E,r}(T) &= \alpha_{E,r}(0) + \alpha_{E,r}(0)QT^2 \\ &\quad - \left[(Q+1)\alpha_{E,r}(0) - (Q-1)\beta_{E,r}(0) \right] T. \end{aligned}$$

For our own use, set

$$a_{E,r} = \frac{\beta_{E,r}(0)}{\alpha_{E,r}(0)}.$$

The Riemann Hypothesis means that

$$\Delta_r = \left[(Q+1)\alpha_{E,r}(0) - (Q-1)\beta_{E,r}(0) \right]^2 - 4\alpha_{E,r}(0)^2 Q < 0,$$

or the same,

$$\begin{aligned} 0 &> \left[(Q+1)\alpha_{E,r}(0) - (Q-1)\beta_{E,r}(0) + 2\alpha_{E,r}(0)\sqrt{Q} \right] \\ &\quad \cdot \left[(Q+1)\alpha_{E,r}(0) - (Q-1)\beta_{E,r}(0) - 2\alpha_{E,r}(0)\sqrt{Q} \right] \\ &= (Q-1)\alpha_{E,r}(0)^2 \left[\sqrt{Q}(1-a_{E,r}) + (1+a_{E,r}) \right] \cdot \left[\sqrt{Q}(1-a_{E,r}) - (1+a_{E,r}) \right]. \end{aligned}$$

Or, equivalently,

$$\begin{cases} \sqrt{Q}(1-a_{E,r}) + (1+a_{E,r}) > 0 \\ \sqrt{Q}(1-a_{E,r}) - (1+a_{E,r}) < 0. \end{cases}$$

That is to say,

$$\frac{\sqrt{Q}-1}{\sqrt{Q}+1} < a_{E,r} < \frac{\sqrt{Q}+1}{\sqrt{Q}-1}.$$

Conjecture 20. (Riemann Hypothesis)

$$1 - \frac{2}{\sqrt{q^r} + 1} < a_{E,r} < 1 + \frac{2}{\sqrt{q^r} - 1}.$$

In particular,

$$a_{E,r} = \frac{\beta_{E,r}(0)}{\beta_{E,r-1}(0)} \rightarrow 1, \quad q \cdot r \rightarrow \infty.$$

Discussion: For $r \rightarrow \infty$, we expect that

$$\beta_{E,r}(0) \sim \widehat{\zeta}_E(1)\widehat{\zeta}_E(2) \cdots \widehat{\zeta}_E(r).$$

So

$$a_{E,r} \sim \widehat{\zeta}_E(r) = 1 + \frac{Nq^{r-1}}{(q^r - 1)(q^{r-1} - 1)}$$

This implies asymptotically the RH holds.

4.2 Distribution of Zeros

To end this paper, let us consider the distribution of zeros under the assumption that the RH holds. Set

$$\cos \theta_{r,p}^E := \frac{(p^r - 1)\beta_{E,r}(0) + (p^r + 1)\alpha_{E,r}(0)}{2\sqrt{p^r}\alpha_{E,r}(0)}$$

Then

$$\cos \theta_{r,p}^E = \frac{(p^r - 1)a_{E,r} + (p^r + 1)}{2\sqrt{p^r}} = \frac{p^r(1 - a_{E,r}) + (1 + a_{E,r})}{2\sqrt{p^r}}.$$

Thus in assuming the Riemann Hypothesis, we have

$$\lim_{pr \rightarrow \infty} \cos \theta_{r,p}^E = 0.$$

Therefore,

$$\lim_{x \rightarrow \infty} \frac{\#\{p \text{ prime} : p \leq x, \alpha \leq \theta_{r,p} \leq \beta\}}{\#\{p \text{ prime} : p \leq x\}} = \int_{\alpha}^{\beta} \delta_{\frac{\pi}{2}} dt,$$

where δ_a denotes the Dirac distribution at a . So the distribution of zeros of high rank zetas are very much different from that of Artin's.

As we expect that, asymptotically,

$$a_{E,r} \sim \widehat{\zeta}_E(r) = 1 + \frac{Np^{r-1}}{(p^r - 1)(p^{r-1} - 1)},$$

for the distributions of zeros of high rank zetas, with $\delta_{\frac{\pi}{2}}$ understood, we should go further to analysis the subdominant term, in order to see the refine structure of the zeros.

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